BRANCH-BASED LOCAL CAPTURE IN TREE-BALL GEOMETRY: SHARP POSITIVE AND NEGATIVE RESULTS

LEVI NEUWIRTH

Abstract. We study a local team-chase problem in a d-regular graph whose ball of large radius around the robber is a tree. We isolate the right local invariant – the deep load along a nonbacktracking path tube – and prove a coordinated package of positive and negative results: (i) sharp counts of geodesic cones and their truncations, (ii) a one-round universal branch-persistence lemma, (iii) a t-round generalization with depth budget $2t+1$, (iv) a sampling barrier showing that any proof relying on the path-tube certificate requires $\Omega((d-1)^{t-1} \cdot t)$ cops, and (v) a finite-order potential-degeneration barrier showing that any local invariant depending only on order-r tube data is strictly insufficient to certify even $(r+1)$-round persistence. Together, these results form a double pincer: certifying t-round persistence by an order-r local invariant requires $r \geq t$, and order-t resolution requires $\Omega((d-1)^{t-1})$ cops to occupy by uniform sampling. This rules out order-r branch-aggregated potentials (for any fixed r) as a route to $\Theta(\log n)$-round chase from polylogarithmic cops, and identifies the precise structural reason why the natural iteration of the one-round argument fails.

Contents

Date: May 6, 2026.

1. Introduction

1.1. **Motivation.** The cops-and-robbers game is a pursuit-evasion game on a graph in which k cops attempt to capture a single robber. The minimum k for which the cops have a winning strategy is the *cop number* $c(G)$. *Meyniel's conjecture* (Frankl, 1987) asserts that $c(G) = O(\sqrt{n})$ for every connected graph on n vertices, and remains the central open problem in the area. The current best upper bound is $c(G) \le n/2^{(1-o(1))\sqrt{\log n}}$ (Lu–Peng; Scott–Sudakov; Frieze–Krivelevich–Loh).

A natural approach toward Meyniel's conjecture, in the spirit of Prałat–Wormald's resolution for random graphs and recent partial progress on expanders and high-girth graphs [2, 5], is to combine random cop placement with a deterministic local chase. The bottleneck of this approach is the *local team-chase problem*: assuming several cops have been placed within a small distance r of the robber, can they coordinate their gradient-descent moves to capture the robber within $O(r \log n)$ rounds?

In high-girth regular graphs, where the local geometry is tree-like, the difficulty of this problem is structural: a single cop is always evaded (Aigner–Fromme), so capture requires multi-cop coordination, and the right coordination invariant is not obvious. In this paper, we work in the cleanest possible local geometry – a d-regular tree-ball – and identify the right invariant, prove that it works for one round, prove that it generalizes to t rounds, and prove a matching barrier.

1.2. **Setup.** Throughout, G denotes a d-regular graph with $d \ge 3$. We fix a vertex $v_0 \in V(G)$ and a radius $R \ge 1$. The fundamental hypothesis is:

Definition 1.1 (Tree-ball)**.** The ball $B_R(v_0)$ is a *tree-ball* if the induced subgraph on $B_R(v_0)$ is a tree. Equivalently, $\mathrm{girth}(G) > 2R$.

In a tree-ball, v_0 has d children (its neighbors), and every interior vertex has exactly one parent and $d-1$ children. We write $S_j(v) = \{x : \mathrm{dist}(x, v) = j\}$ for the depth-j shell at v.

Definition 1.2 (Geodesic cone)**.** For $v \in V(G)$ with $B_r(v)$ a tree-ball and $u \in N(v)$, the *geodesic cone* through u at radius r is

$$C_u(v, r) := \{x \in B_r(v) \setminus \{v\} : \text{the unique geodesic from } x \text{ to } v \text{ has penultimate vertex } u\}.$$

The *truncated cone at depth* $\ge k$ is

$$C_u^{\ge k}(v, r) := \{x \in C_u(v, r) : \mathrm{dist}(x, v) \ge k\}.$$

Definition 1.3 (Path tube)**.** For $v_0 \in V(G)$ with $B_R(v_0)$ a tree-ball, and a nonbacktracking path $\sigma = (v_0, v_1, \ldots, v_t)$ from v_0, and a depth threshold k, the *t-tube at depth* $\ge k$ is

$$T_\sigma^{\ge k}(R) := \big\{x \in B_R(v_0) : \mathrm{dist}(x, v_0) \ge k, \\ \text{the unique geodesic from } x \text{ to } v_0 \text{ visits } v_t, v_{t-1}, \ldots, v_1 \text{ in order}\big\}.$$

The cops execute *gradient descent*: at each cop turn, every cop moves to a neighbor minimizing distance to the current robber position, breaking ties by an arbitrary rule (deterministic or randomized). The order of play is: cops move, then robber moves. A robber move to $w \in N(v(t))$ is *safe* if no cop occupies w at the start of the robber's turn (i.e., after the cops' move).

For the robber's position v and a neighbor $u' \in N(v)$, we say a cop c *contributes to branch u' at v* if $c \ne v$ and the geodesic from c to v has penultimate vertex u'. The *branch-load support* at v is the set of $u' \in N(v)$ that receive at least one contributing cop.

1.3. **Results.** The paper has four main results, prefaced by a sharp counting lemma.

Lemma 1.4 (Sharp shell and cone counts)**.** *Let G be d-regular with $d \geq 3$, and suppose $B_r(v)$ is a tree-ball. Then for every $u \in N(v)$ and $1 \leq j \leq r$,*

$$|C_u(v,r) \cap S_j(v)| = (d-1)^{j-1}, \qquad |S_j(v)| = d(d-1)^{j-1}.$$

Hence

$$|C_u(v,r)| = \sum_{j=1}^{r}(d-1)^{j-1} = \frac{(d-1)^r - 1}{d-2}, \quad |B_r(v)| = 1 + d \cdot \frac{(d-1)^r - 1}{d-2}.$$

For the truncated cone,

$$|C_u^{\geq k}(v,r)| = \sum_{j=k}^{r}(d-1)^{j-1}, \quad 1 \leq k \leq r.$$

The first main result is a probabilistic deep-load corollary.

Corollary 1.5 (Deep-load occupancy)**.** *Let G be d-regular with $B_r(v)$ a tree-ball, and let $X_1, \dots, X_m$ be i.i.d. uniform samples from $B_r(v)$. Set*

$$p_{d,r}^{\geq k} := \frac{|C_u^{\geq k}(v,r)|}{|B_r(v)|},$$

which by Lemma 1.4 does not depend on the choice of $u \in N(v)$, since all branches in a tree-ball have equal size. Then

$$\Pr[\exists u \in N(v) : C_u^{\geq k}(v,r) \text{ receives no sample}] \leq d \cdot (1 - p_{d,r}^{\geq k})^m.$$

The second is the universal t-round persistence theorem with the correct depth budget.

Theorem 1.6 (Universal t-round deep-load persistence)**.** *Let G be d-regular with $d \geq 3$, $v_0 \in V(G)$, and assume $B_R(v_0)$ is a tree-ball for some $R \geq 2t+1$. Suppose that for every nonbacktracking path $\sigma = (v_0, v_1, \dots, v_t)$ of length t from v_0, there exists at least one cop in $T_\sigma^{\geq 2t+1}(R)$.*

Then for every nonbacktracking robber path $(v_0, v_1, \dots, v_t)$ and every $s \in \{1, \dots, t\}$, after the cops' gradient-descent step at round s, exactly one of the following holds:

- *(a) a cop occupies v_s, so the robber's intended move to v_s is illegal; or*
- *(b) the move to v_s is legal, and after the robber moves to v_s, the branch-load support at v_s has size at least 2.*

The third is the sampling barrier.

Theorem 1.7 (Barrier on branch-tube certificates)**.** *Fix $d \geq 3$. In the d-regular tree-ball model, any argument that places m cops by uniform sampling from $B_R(v_0)$ and relies on occupancy of every length-t deep tube $T_\sigma^{\geq 2t+1}(R)$ requires*

$$m = \Omega\big((d-1)^{t-1} \cdot t\big)$$

cops. In particular, for $m = \mathrm{polylog}(n)$ and fixed d, this method certifies only $t = O(\log\log n)$ rounds, not $t = \Theta(\log n)$.

The fourth is a complementary degeneration barrier showing that bounded-order compressions of the path-tube data are themselves insufficient. Theorem 4.6 (stated and proved in Section 4.5) constructs, for every $r \geq 1$, two cop configurations X and Y that agree on all order-r tube data but differ in their post-$(r+1)$-round branch-load support: X has support ≥ 2 while Y has support exactly 1. Hence no certificate depending only on order-r data can certify universal $(r+1)$-round persistence.

1.4. **Strategic significance.** Theorems 1.6, 1.7, and 4.6 together capture the precise reach of branch-tube methods. The natural iteration of the one-round argument extends to t rounds with the correct depth budget $2t+1$ (Theorem 1.6), but the path-entropy cost is exponential in t (Theorem 1.7), and any compression to bounded-order data fails dynamically (Theorem 4.6). Together, the two barriers tighten this trade-off: certifying t-round persistence by an order-r local invariant requires $r \geq t$, and order-t resolution requires $\Omega((d-1)^{t-1})$ cops to occupy.

The barriers rule out a substantial family of proof techniques but are silent on what could replace them. The pivots in Section 5 sketch the most natural escape routes: epoch refresh (which probably needs a global progress measure to succeed), soft potentials that necessarily break locality or symmetry, and a non-local spectral approach via nonbacktracking-walk concentration.

2. Sharp Counts

2.1. Proof of Lemma 1.4.

Proof. Since $B_r(v)$ is a tree-ball and G is d-regular, v has exactly d children in the rooted tree (one per neighbor in $N(v)$), and every non-root vertex at depth $< r$ has exactly $d-1$ children. Fix $u \in N(v)$. The branch rooted at u is a $(d-1)$-ary rooted tree of depth $r-1$; its level $j-1$ has $(d-1)^{j-1}$ vertices, so the depth-j shell of the branch (relative to v) has $(d-1)^{j-1}$ vertices.

Hence $|C_u(v,r) \cap S_j(v)| = (d-1)^{j-1}$ for $1 \leq j \leq r$. Summing over the d branches gives $|S_j(v)| = d(d-1)^{j-1}$, and summing j from 1 to r gives the cone size. Including v itself yields the ball size, and restricting $j \geq k$ yields the truncated cone count. □

Remark 2.1. The ratio $|C_u^{\geq k}(v,r)|/|B_r(v)|$ tends to $1/d$ as $r \to \infty$ for fixed k. Thus each branch asymptotically owns a $1/d$ fraction of the ball, and depth-k truncation costs only $O(d^k)$ vertices out of $\Theta(d^r)$, which is negligible at large r.

2.2. Proof of Corollary 1.5.

Proof. For a fixed branch $u \in N(v)$, the probability that no X_i lies in $C_u^{\geq k}(v,r)$ is $(1-p_{d,r}^{\geq k})^m$. Union bounding over the d choices of u gives the claim. □

For $k=3$ and r large, $p_{d,r}^{\geq 3} \to 1/d - O(d^{-r})$, so $m = \Theta(d \log d)$ samples suffice to hit every truncated branch with high probability. For fixed d, $m = O(\log n)$ drives the failure probability polynomially small in n.

3. One-Round and t-Round Persistence

3.1. **The one-round case.** We first prove the case $t=1$ separately, both for clarity and because the argument is the model for the multi-round induction. Lemma 3.1 below is the $t=1$ specialization of Theorem 1.6, presented here in self-contained form.

Lemma 3.1 (One-round persistence)**.** *Let G be d-regular with $d \geq 3$, $v_0 \in V(G)$, and assume $B_3(v_0)$ is a tree-ball. Suppose that for every $u \in N(v_0)$, there is at least one cop in $C_u^{\geq 3}(v_0, R)$ for some $R \geq 3$.*

Then for every $w \in N(v_0)$, after the cop gradient-descent step, exactly one of the following holds:

(a) a cop occupies w, so the robber's intended move to w is illegal; or

(b) the move to w is legal, and after the robber moves to w, the branch-load support at w has size at least 2.

Proof. Fix $w \in N(v_0)$. After the cop gradient-descent step, if some cop occupies w then case (a) holds and we are done. Otherwise we exhibit both a descendant witness and a parent witness at w, establishing case (b).

Descendant witness. By hypothesis there is a cop $c^{\downarrow} \in C_w^{\geq 3}(v_0, R)$. Then $\mathrm{dist}(c^{\downarrow}, v_0) = j \geq 3$ with the geodesic from $c^{\downarrow}$ to v_0 passing through w. Gradient descent toward v_0 moves $c^{\downarrow}$ to its parent in the tree rooted at v_0, which is at depth $j-1 \geq 2$, still in the w-subtree. After this move, $c^{\downarrow}$ is at distance $j-1-1 = j-2 \geq 1$ from w. The geodesic from this new position to w passes through some child of w in the tree, so $c^{\downarrow}$ contributes to a descendant branch at w.

Parent witness. Pick $u \in N(v_0)$ with $u \neq w$ (possible since $d \geq 3 \geq 2$). By hypothesis there is a cop $c^{\uparrow} \in C_u^{\geq 3}(v_0, R)$. After gradient descent, $c^{\uparrow}$ remains in the u-subtree at depth ≥ 2 from v_0. Relative to w, the geodesic from $c^{\uparrow}$ to w passes through v_0 (the unique tree path between distinct subtrees), so $c^{\uparrow}$ contributes to the parent branch at w through v_0.

Combining witnesses. The descendant and parent witnesses lie in distinct branches at w, so the branch-load support at w has size at least 2, establishing case (b). ☐

3.2. **Proof of Theorem 1.6.** We now extend the one-round argument to general t, with depth budget $2t+1$. The reason this budget is necessary, not merely sufficient, is illustrated in the sharpness remark of Section 3.3.

Proof of Theorem 1.6. Fix any nonbacktracking robber path $(v_0, v_1, \ldots, v_t)$ and any $s \in \{1, \ldots, t\}$. We analyze round s on the assumption that no cop landed on $v_{s'}$ during the round-s' gradient-descent step, for each $1 \leq s' < s$; otherwise case (a) holds at the first such s', establishing the dichotomy there. Under this assumption the robber is at v_{s-1} at the start of round s. After the cops' gradient-descent step, if some cop occupies v_s then case (a) holds and we are done. Otherwise we exhibit both a descendant-branch witness and a parent-branch witness at v_s, establishing case (b).

Descendant witness at v_s. Let $\sigma_s = (v_0, v_1, \ldots, v_s)$ be the robber's trajectory through time s. Extend σ_s to any nonbacktracking length-t path σ (e.g., by appending $t-s$ steps in any nonbacktracking way). By hypothesis there is a cop $c_\sigma^{\downarrow} \in T_\sigma^{\geq 2t+1}(R)$ with initial depth $j_0 \geq 2t+1$, and the unique geodesic from $c_\sigma^{\downarrow}$ to v_0 visits $v_t, v_{t-1}, \ldots, v_1$ in order.

In a tree-ball, gradient descent toward any vertex on the cop's geodesic to v_0 is forced: the unique distance-decreasing neighbor is the cop's parent in the v_0-rooted tree. Therefore each round the cop moves exactly one step upward along its geodesic to v_0. After s rounds, the cop has moved s steps up its geodesic to v_0, so its current geodesic to v_0 visits $v_s, v_{s-1}, \ldots, v_1$ in order, and its depth from v_0 is $j_0 - s$.

Since $j_0 \geq 2t+1$ and $s \leq t$, we have $j_0 - s \geq t+1 > s$, so the cop remains *strictly below* v_s: it is in the v_s-subtree at depth $\geq s+1$ from v_0, equivalently at distance ≥ 1 from v_s within that subtree. Hence the geodesic from the cop's current position to v_s has penultimate vertex equal to some child of v_s, and the cop contributes to a descendant branch at v_s.

Parent witness at v_s. Choose any neighbor $u_0 \in N(v_0)$ with $u_0 \neq v_1$ (possible since $d \geq 3$). Extend u_0 to any nonbacktracking length-t path $\sigma' = (v_0, u_0, \ldots)$. By hypothesis there is a cop $c^{\uparrow} \in T_{\sigma'}^{\geq 2t+1}(R)$.

Initially $c^{\uparrow}$ is in the u_0-subtree at depth $\geq 2t+1$. At round 1, gradient descent toward v_0 moves $c^{\uparrow}$ to depth $2t$, still in the u_0-subtree. At round 2, gradient descent toward v_1 has target in a different subtree from $c^{\uparrow}$, so the cop's geodesic to v_1 goes up through v_0. The unique distance-decreasing neighbor is the cop's parent in the v_0-rooted tree, so the cop moves to depth $2t-1$, still in the u_0-subtree. The same argument applies at every round k with $1 \leq k \leq s$: from the cop's position in the u_0-subtree at depth $2t+2-k$, the robber's current position v_{k-1} is reached only via v_0 (since $u_0 \neq v_1$), so the unique distance-decreasing direction is up to the parent.

By induction, after s rounds, $c^{\uparrow}$ is at depth $2t+1-s$ from v_0, still in the u_0-subtree. Since $u_0 \neq v_1$, the entire u_0-subtree is disjoint from the v_1-subtree, and since each v_j for $j \geq 1$ lies in the v_1-subtree, the cop $c^{\uparrow}$ at depth $2t+1-s \geq t+1 \geq 1$ from v_0 in the u_0-subtree is on the parent side of v_s. For $s \geq 1$, the geodesic from $c^{\uparrow}$ to v_s passes through v_0, so the penultimate vertex of that geodesic (the one adjacent to v_s) is v_{s-1}. So $c^{\uparrow}$ contributes to the parent branch of v_s through v_{s-1}.

Combining witnesses. The descendant witness contributes to a non-parent branch at v_s, and the parent witness contributes to the parent branch at v_s. These are distinct branches, so the branch-load support at v_s is at least 2, establishing case (b). □

3.3. **The depth budget is sharp.** The hypothesis $j_0 \geq 2t+1$ in Theorem 1.6 is necessary, not merely sufficient. We illustrate.

Remark 3.2 (Sharpness of the depth budget)**.** Consider $j_0 = 2t$. After $s = t$ rounds, the cop is at depth $j_0 - t = t$ from v_0, with the robber at v_t (also depth t). The cop's geodesic to v_0 contains $v_t, v_{t-1}, \ldots, v_1$, so the cop is at v_t itself, not at a strict descendant. This blocks the robber's move (case (a)) but does not provide a descendant branch at v_t for the post-move analysis. Reducing further to $j_0 < 2t$ would place the cop on an ancestor of v_t, and the cop would contribute to the parent branch only – destroying the support-2 claim.

4. The Barrier

4.1. **Tube density.**

Lemma 4.1 (Asymptotic tube density)**.** *For fixed $d \geq 3$ and a length-t nonbacktracking prefix σ in a d-regular tree-ball $B_R(v_0)$ with $R \geq 2t+1$,*

$$|T_\sigma^{\geq 2t+1}(R)| = \sum_{j=2t+1}^{R} (d-1)^{j-t} = \sum_{\ell=t+1}^{R-t} (d-1)^{\ell} = (d-1)^{t+1} \cdot \frac{(d-1)^{R-2t}-1}{d-2}.$$

The tube density

$$q_t(R) := \frac{|T_\sigma^{\geq 2t+1}(R)|}{|B_R(v_0)|}$$

satisfies, as $R \to \infty$,

$$q_t(R) \to \frac{1}{d(d-1)^{t-1}}.$$

Proof. Each vertex in $T_\sigma^{\geq 2t+1}(R) \cap S_j(v_0)$ is a descendant of v_t at depth j from v_0, hence at depth $j-t$ within the v_t-rooted subtree. The latter is a $(d-1)$-ary tree (since v_t has $d-1$ children in the tree-ball), so its depth-$(j-t)$ shell has $(d-1)^{j-t}$ vertices for $j \geq t+1$. Restricting to $j \geq 2t+1$, summing, and substituting $\ell = j - t$ gives the closed form above.

For the asymptotic density: both $|T_\sigma^{\geq 2t+1}(R)|$ and $|B_R(v_0)|$ are geometric sums dominated by their top shells. The top shell of the tube at $j = R$ has $(d-1)^{R-t}$ vertices; the top shell of the ball at $j = R$ has $d(d-1)^{R-1}$ vertices. Hence the ratio tends to

$$\frac{(d-1)^{R-t}}{d(d-1)^{R-1}} = \frac{1}{d(d-1)^{t-1}},$$

as claimed. □

4.2. **Number of length-t prefixes.**

Lemma 4.2. *The number of nonbacktracking length-t paths from v_0 in a d-regular tree-ball is*

$$N_t = d(d-1)^{t-1}.$$

Proof. The first step has d choices (any neighbor of v_0). Each subsequent step has $d-1$ choices (any neighbor of the current vertex except the immediately previous one). So $N_t = d(d-1)^{t-1}$. □

4.3. **Proof of Theorem 1.7.**

Proof of Theorem 1.7. Suppose m cops are sampled i.i.d. uniformly from $B_R(v_0)$, and the proof relies on every length-t tube being occupied. By Lemma 4.1, for each fixed prefix σ,

$$\Pr\bigl[T_\sigma^{\geq 2t+1}(R) \text{ is empty}\bigr] = (1 - q_t(R))^m \leq \exp(-m \cdot q_t(R)).$$

Union bounding over the N_t prefixes,

$$\Pr[\exists \sigma : T_\sigma \text{ empty}] \leq N_t \cdot \exp(-m \cdot q_t(R)).$$

For this to be $o(1)$, we require

$$m \geq \frac{\log N_t + \omega(1)}{q_t(R)}.$$

Using $N_t = d(d-1)^{t-1}$ and $q_t(R) \sim 1/[d(d-1)^{t-1}]$ from Lemmas 4.2 and 4.1,

$$m \gtrsim d(d-1)^{t-1} \cdot \log\bigl[d(d-1)^{t-1}\bigr].$$

For fixed d, this is $m = \Omega((d-1)^{t-1} \cdot t)$.

Inverting. Suppose $m = \operatorname{polylog}(n)$. From $(d-1)^{t-1} \leq m$, taking logarithms gives

$$t - 1 \leq \frac{\log m}{\log(d-1)} = \frac{O(\log\log n)}{\log(d-1)},$$

so $t = O(\log\log n)$.

In particular, the certificate cannot certify $t = \Theta(\log n)$ rounds with $m = \operatorname{polylog}(n)$ cops. To certify $t = \Theta(\log n)$ rounds, the certificate requires $m = (d-1)^{\Theta(\log n)} \cdot \Theta(\log n) = n^{\Theta(\log(d-1))} \cdot \Theta(\log n)$ cops, which is polynomial in n with exponent depending on d. □

4.4. **What the barrier says, precisely.** The barrier rules out a specific proof technique: it does not say that the $\Theta(\log n)$-round local team-chase is impossible, only that it cannot be obtained by occupying every length-t deep tube uniformly. The barrier identifies the cost as exponential in t, with the exponent base $d - 1$ – the local branching factor of the tree-ball.

Remark 4.3. The barrier is robust to mild relaxations of the certificate. For example, requiring *most* (rather than all) tubes to be occupied still requires $\Omega((d-1)^{t-1})$ cops in expectation: the expected number of empty tubes is $N_t(1 - q_t)^m$, and bounding this even by 1 gives $m \geq \log N_t/q_t$, which is the same order. Similarly, weakening the depth threshold from $2t + 1$ to any $\Theta(t)$ does not change the exponential dependence on t, only the constant in the exponent.

4.5. **A finite-order potential-degeneration barrier.** Theorem 1.7 shows that occupying every length-t deep tube is exponentially expensive in t under uniform local sampling. A natural response is to compress the local state: instead of tracking every length-t tube, one might try to use an invariant that remembers only bounded-order tube data, aggregating all information below a fixed depth. The next theorem shows that this cannot certify persistence beyond one additional round.

Definition 4.4 (Order-r tube profile)**.** Fix a rooted tree-ball $B_R(v_0)$ in a d-regular graph. For a cop configuration X and an integer $r \geq 1$, the *order-r tube profile* of X is the family

$$\Pi_r(X) := \bigl\{N_X(\sigma, j)\bigr\}_{\sigma, j},$$

where $\sigma = (v_0, v_1, \ldots, v_r)$ ranges over all nonbacktracking paths of length r from v_0, and $j \geq r + 1$, and

$$N_X(\sigma, j) := \#\bigl\{x \in X : \operatorname{dist}(x, v_0) = j, \text{ and the unique geodesic from } x \text{ to } v_0 \text{ begins with } \sigma\bigr\}.$$

Thus $\Pi_r(X)$ records the exact depth histogram inside every length-r tube, but forgets how the mass in that tube splits below level r.

Definition 4.5 (Order-r invariant)**.** A local invariant F on cop configurations in a rooted tree-ball is *order-r* if it factors through Π_r, i.e. if $F(X) = F(Y)$ whenever $\Pi_r(X) = \Pi_r(Y)$. It is *neighbor-symmetric* if it is invariant under rooted automorphisms of the tree-ball fixing v_0.

Theorem 4.6 (Finite-order potential-degeneration barrier)**.** *Fix $d \geq 3$ and $r \geq 1$. Let $B_R(v_0)$ be a d-regular tree-ball with $R \geq 2r+3$. Then there exist two cop configurations X and Y in $B_R(v_0)$ such that:*

(i) $\Pi_r(X) = \Pi_r(Y)$*; in particular, every order-r invariant takes the same value on X and Y.*
(ii) There exists a nonbacktracking path

$$(v_0, v_1, \ldots, v_{r+1})$$

such that this robber path is safe in both configurations, but after $r+1$ rounds of cop gradient descent followed by robber motion along that path:
- *in configuration X, the branch-load support at v_{r+1} has size at least 2;*
- *in configuration Y, the branch-load support at v_{r+1} has size exactly 1.*

Consequently, no certificate whose hypotheses depend only on order-r data can correctly distinguish ≥ 2-branch support from 1-branch support after $r+1$ rounds.

Proof. Fix any nonbacktracking path

$$(v_0, v_1, \ldots, v_{r+1})$$

inside the tree-ball. Since $d \geq 3$, the vertex v_{r+1} has at least two distinct children in the rooted tree. Choose two such children and call them a and b.

We now define two cop configurations, each consisting of exactly two cops, both at depth $2r+3$ from v_0.

Configuration X. Place one cop in the subtree rooted at a and one cop in the subtree rooted at b, both at total depth $2r+3$ from v_0.

Configuration Y. Place both cops in the subtree rooted at a, at distinct vertices, again at total depth $2r+3$ from v_0. (Since the depth-$(r+1)$ shell of the a-subtree has $(d-1)^{r+1} \geq 2$ vertices, this is possible.)

In both configurations, every cop lies in the same length-r tube determined by the prefix

$$\sigma_0 = (v_0, v_1, \ldots, v_r),$$

and every cop lies at the same total depth $2r+3$ from v_0. Therefore the order-r tube counts are identical:

$$N_X(\sigma_0, 2r+3) = N_Y(\sigma_0, 2r+3) = 2,$$

and $N_X(\sigma, j) = N_Y(\sigma, j) = 0$ for all other (σ, j). Hence $\Pi_r(X) = \Pi_r(Y)$. The order-r invariant cannot see how the two cops in σ_0 split between the children a and b of v_{r+1}.

Now consider the robber path

$$(v_0, v_1, \ldots, v_{r+1}).$$

We claim it is safe in both configurations. Each cop starts at depth $2r+3$ from v_0. In a tree-ball, gradient descent toward any robber position on the cop's geodesic to v_0 is forced: each round the cop moves exactly one step upward toward v_0. Hence after s rounds, each cop is at depth $2r+3-s$ from v_0. After the cop's move at round s, the robber moves $v_{s-1} \to v_s$. The cop's distance to the robber's destination v_s is

$$(2r+3-s) - s = 2r+3-2s.$$

For every $1 \leq s \leq r+1$ this is at least 1, so no cop ever lands on v_s during these $r+1$ rounds. Thus the robber path is safe in both configurations.

Finally, inspect the cop locations after $r+1$ rounds. Each cop has moved upward by exactly $r+1$ steps, so each is now at depth

$$2r + 3 - (r + 1) = r + 2$$

from v_0, while v_{r+1} has depth $r+1$. Hence each cop is now a child of v_{r+1}.

In configuration X, the two cops lie on two distinct children of v_{r+1}, namely a and b. Therefore the branch-load support at v_{r+1} has size at least 2.

In configuration Y, both cops climbed within the a-subtree and now sit on its root, namely a itself (since gradient descent in a tree-ball is forced upward). Therefore both cops contribute to the same branch at v_{r+1}, and the branch-load support has size exactly 1.

This proves (ii). Since $\Pi_r(X) = \Pi_r(Y)$ but the outcomes differ, no certificate depending only on order-r data can correctly classify both configurations. □

Corollary 4.7. *Any invariant depending only on first-level branch data, even with exact depth histograms inside each first-level branch, cannot certify universal* 2*-round persistence in the tree-ball model.*

Proof. Take $r = 1$ in Theorem 4.6. □

Remark 4.8. Theorems 1.7 and 4.6 are complementary. Theorem 1.7 shows that full order-t path information is sufficient but exponentially expensive to certify by uniform random local sampling. Theorem 4.6 shows that any bounded-order compression of that information is dynamically insufficient beyond one additional round. Together they imply that, in the tree-ball model, there is no bounded-order shortcut around path entropy: certifying t-round persistence by a local invariant requires at least order-t resolution, and order-t resolution requires $\Omega((d-1)^{t-1})$ cops to occupy. Together they rule out order-r branch-aggregated potentials for every fixed r as a route to $\Theta(\log n)$-round chase from polylogarithmic cops.

5. Discussion: Pivots and Open Directions

The barriers of Theorems 1.7 and 4.6 together rule out a substantial family of approaches but leave several escape routes open.

5.1. Pivot A: Epoch refresh.

Run epochs of length $t_0 = O(\log \log n)$, for which the tube certificate is affordable with polylog(n) cops, then re-randomize and restart. The barrier does not preclude this: it constrains a single epoch but not a sequence of epochs.

The crucial open question is whether one t_0-round epoch produces *permanent* progress (a coarser robber freedom parameter shrinks by a constant factor). In a tree-ball, no such parameter is obvious: the robber's territory is unbounded by hypothesis, and after one epoch the robber can simply retreat to a fresh subtree. This suggests that epoch refresh in the pure tree-ball model is unlikely to succeed, and the program must combine the local tree-ball analysis with a global structural bound that limits the robber's total territory.

5.2. Pivot B: Soft potentials must break locality or symmetry.

Theorem 4.6 rules out any *order-r* potential as a route to $(r+1)$-round persistence. To escape the barrier via a soft potential, the potential must therefore depend on at least one of:

- *Unbounded local order:* information that depends on tube data of order growing with t. This re-introduces path-entropy costs and hits Theorem 1.7.
- *Non-local data:* information that depends on the global graph structure beyond the local tree-ball, such as cycle structure at radius $> R$, or spectral data of the full graph.
- *Asymmetric data:* information that breaks the rooted-tree symmetry, such as a fixed orientation, an external labeling, or a privileged subset of vertices not invariant under tree automorphisms.

The most promising candidate among these is the second: *nonbacktracking-walk concentration* of cop mass on the full graph. If cops are placed by a global averaging that respects the nonbacktracking spectrum, the resulting cop distribution at time t may concentrate on the robber's location at exponential rate $\rho(B)^{-1}$, where B is the Hashimoto nonbacktracking transition matrix of the full graph. The local tree-ball view of such a placement is no longer order-r for any fixed r; it inherits global spectral information that is invisible to any local invariant. This suggests that purely local tree-ball analysis is fundamentally insufficient.

5.3. **Pivot C: Hierarchical packets.** Use cop packets at multiple depth scales. A packet at scale j guarantees persistence for 2^j rounds inside a tube of depth $\Theta(2^j)$. Packets are activated sequentially, with later packets covering longer ranges. This is a multiscale variant of refresh.

A naive cost analysis is discouraging: each scale costs $(d-1)^{2^j}$ cops by the barrier of Theorem 1.7, and at the top scale $j = \log\log n$ this is already $(d-1)^{\log n} = n^{\log_2(d-1)}$, which is polynomial in n. So the straightforward hierarchical-packet scheme does not deliver polylogarithmic cop count. This matches the cost of Theorem 1.7 inverted at $t = \Theta(\log n)$: the multi-scale scheme inherits the same fundamental cost as the single-scale scheme it tries to circumvent. To salvage the idea one would need a sub-exponential variant in which packets at scale j exploit structure (epoch refresh between scales, partial coverage of tubes, or shared cops across scales) to evade the barrier. We do not pursue this here.

5.4. **Beyond the tree-ball model.** The deepest limitation of the present results is the tree-ball hypothesis. Real expanders – including Ramanujan graphs of girth $\Theta(\log n)$ – are tree-balls only up to radius $\Theta(\log n)$, but a chase argument typically needs to reason about radii $r \approx \log n/\gamma$ where γ is the spectral gap. For γ bounded below by a constant, these match up to constants; but for γ small, the chase exits the tree-ball and must contend with cycles. In the cyclic regime, the geodesic cone $C_u(v,r)$ is no longer a clean object (vertices may have multiple shortest paths to v), and the entire branch decomposition becomes ill-defined.

A full local team-chase theorem on expanders likely requires either (i) a generalization of the present analysis to graphs with bounded but nontrivial girth, where "tubes" become equivalence classes of approximately-tree-like geodesics, or (ii) a fundamentally different invariant – spectral, fence-based, or potential-theoretic – that does not rely on tree branch decomposition at all.

6. Conclusion

We have proved a sharp local persistence theorem in the tree-ball model and two matching barriers: a sampling barrier showing that the natural iteration of the one-round argument is exponentially expensive in time, and a finite-order potential-degeneration barrier showing that any bounded-order compression of the path-tube certificate is dynamically insufficient. Together they tighten the trade-off: certifying t-round persistence by an order-r local invariant requires $r \geq t$, and order-t resolution requires $\Omega((d-1)^{t-1})$ cops to occupy by uniform sampling. Together, they rule out order-r branch-aggregated potentials (for any fixed r) as a route to $\Theta(\log n)$-round chase from polylogarithmic cops in the tree-ball model. A structural new ingredient – non-local information, broken symmetry, or escape from the tree-ball regime entirely – is required.

The four main results – deep-load occupancy, t-round persistence with depth budget $2t+1$, the sampling barrier, and the finite-order potential-degeneration barrier – together with the supporting tube-counting lemma form a self-contained module that we hope will be useful as one block in any future proof of Meyniel's conjecture via local chase arguments. The pivots in Section 5 suggest several directions in which the program could be continued, though we leave their analysis to future work.

Acknowledgments

I warmly thank Eric Jovinelly, who led me through Graph Theory at Brown and greatly assisted me in building both graph theoretic and broader mathematical skills and intuition. I also thank my peers in the Spring 2026 S02 of Graph Theory at Brown.

References

[1] M. Aigner and M. Fromme, *A game of cops and robbers*, Discrete Applied Mathematics, vol. 8, no. 1, 1984, pp. 1–12.

[2] P. Bradshaw, S. A. Hosseini, B. Mohar, and L. Stacho, *On the cop number of graphs of high girth*, Journal of Graph Theory, 2023.

[3] P. Frankl, *Cops and robbers in graphs with large girth and Cayley graphs*, Discrete Applied Mathematics, vol. 17, 1987, pp. 301–305.

[4] A. Frieze, M. Krivelevich, and P.-S. Loh, *Variations on cops and robbers*, Journal of Graph Theory, vol. 69, no. 4, 2012, pp. 383–402.

[5] S. A. Hosseini, B. Mohar, and S. Gonzalez Hermosillo de la Maza, *Meyniel's conjecture on graphs of bounded degree*, Journal of Graph Theory, vol. 97, 2021, pp. 401–407.

[6] L. Lu and X. Peng, *On Meyniel's conjecture of the cop number*, Journal of Graph Theory, vol. 71, no. 2, 2012, pp. 192–205.

[7] R. Nowakowski and P. Winkler, *Vertex-to-vertex pursuit in a graph*, Discrete Mathematics, vol. 43, 1983, pp. 235–239.

[8] P. Prałat and N. Wormald, *Meyniel's conjecture holds for random graphs*, Random Structures & Algorithms, vol. 48, no. 2, 2016, pp. 396–421.

[9] A. Quilliot, *Jeux et points fixes sur les graphes*, Thèse de 3ème cycle, Université de Paris VI, 1978.

[10] A. Scott and B. Sudakov, *A bound for the cops and robbers problem*, SIAM Journal on Discrete Mathematics, vol. 25, no. 3, 2011, pp. 1438–1442.

Brown University