BALL-OCCUPATION CERTIFICATES UNDER COARSE GRAPH PROJECTIONS
DEGREE REDUCTION, SQUARE-ROOT HARD FAMILIES, AND TOROIDAL BARRIERS

LEVI NEUWIRTH

Abstract. We isolate an abstract strategy-transfer principle for Cops and Robber. Let $\pi : V(H) \to V(G)$ have fibers of order at most P, and suppose that for a scale factor $\lambda \geq 1$ the distances between distinct fibers satisfy

$$\lambda(\mathrm{dist}_G(u,v) - 1) + 1 \leq \mathrm{dist}_H(x,y) \leq \lambda(\mathrm{dist}_G(u,v) + 2) - 2,$$

while every fiber has diameter at most $2(\lambda - 1)$. If the N-vertex base graph has uniform multi-source growth at the square-root scale, then a Hall assignment occupies a lifted macro-ball before the robber can leave it, giving

$$c(H) \leq CP\big(d^3 + \log(ePN)\big)\sqrt{N}$$

and capture time at most λR. The theorem uses only the displayed projection properties, not the internal form of the fibers.

Iterated degree reduction of Hosseini–Mohar–Gonzalez Hermosillo de la Maza has exactly this geometry, with $\lambda = 3^k$. Applied to $G(N,p)$ of expected degree $(\log N)^4$, it yields connected subcubic graphs of order M with

$$h(H) \geq (\log M)^{-O(1)}, \qquad M^{\frac{1}{2} - \frac{9+o(1)}{2\log\log\log M}} \leq c(H) \leq \sqrt{M}\,(\log M)^{O(1)}.$$

Thus the known stressing family has the square-root exponent, with a polylogarithmically tight upper bound; the lower convergence to $1/2$ is only triple-logarithmic.

Finally, we prove a sharp limitation of the strategy class. If a common prepositioned bank must, after learning a robber start v, match distinct cops to every vertex of $B(v,R)$ using travel distance at most R, then

$$\mathrm{Occ}_R(G) \geq \frac{\sum_v |B(v,R)|}{\max_x |B(x,2R)|}.$$

For every fixed $k \geq 2$, the Cartesian torus $C_L^{\square k}$ has $|V| = L^k$, vertex expansion $\Theta_k(L^{-1})$, exact cop number $k+1$, and $\mathrm{Occ}_R = \Omega_k(L^k)$ for every radius R. Hence, for every $\delta > 0$, there are bounded-degree graphs with $h(G) \geq |G|^{-\delta}$ and constant cop number on which one-shot occupation still costs a linear number of cops. A cubic four-cycle replacement gives a degree-three instance at exponent $\delta = 1/2$. Any universal robustness theorem must therefore use adaptive multi-round pursuit, cop reuse, or another strategy not reducible to one-shot occupation.

1. Introduction and main conclusions

The multi-cop version of Cops and Robber was developed by Aigner and Fromme, who proved that three cops suffice on every planar graph [1]. Meyniel's conjecture asks whether every connected n-vertex graph has cop number $O(\sqrt{n})$. The current best universal upper bound remains

$$\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}},$$

Date: July 2026.

2020 *Mathematics Subject Classification.* 05C57, 05C40, 05C12.

Key words and phrases. Cops and Robber; Meyniel's conjecture; graph products; degree reduction; coarse graph projections; expansion; occupation certificates.

proved independently by Lu–Peng and Scott–Sudakov [9, 13]. Bose–Esperet–Hodor–Joret–Micek–Rambaud recently extended the same scale of bound from graph order to vertex-cover number [2]. Expansion is one of the principal settings in which polynomial savings are known: Bradshaw–Hosseini–Mohar–Stacho obtain weak Meyniel bounds from bounded-degree expansion restricted to sublinear set scales [3], while Clow's withdrawn preprint developed a closely related structural program connecting failure of weak Meyniel to high-cop expanding examples [6].

The motivation here is the effect of bounded-degree replacement gadgets on pursuit. The degree-reduction construction of Hosseini–Mohar–Gonzalez Hermosillo de la Maza (HMGHM) preserves lower bounds on cop number and produces subcubic graphs with cop number $M^{1/2-o(1)}$ [7]. A natural converse question is whether a useful upper strategy on the base graph survives the replacement tower.

An arbitrary winning strategy does not lift transparently. Moving one step in the quotient may require a squad dispersed through a cloud to reorganize while the robber continues moving. The successful object is narrower and more stable: an *occupation certificate* that assigns distinct cops to all vertices of a region before the robber can leave it. Distance stretching slows both deployment and escape, and a bounded normalized additive error leaves a strict timing margin.

The first result is therefore stated for an abstract projection, rather than for the HMGHM gadget. The gadget enters only later, through an exact metric calculation. The resulting upper bound is stronger quantitatively than the notation $M^{1/2+o(1)}$ suggests: it is $\sqrt{M}$ times a polylogarithmic factor. By contrast, the available lower bound approaches the square-root exponent at the triple-logarithmic rate displayed in the abstract. These two facts should not be conflated merely because both can be written $M^{1/2+o(1)}$.

The final result marks the boundary of the mechanism. A one-shot occupation certificate needs polynomial ball amplification between radii R and $2R$. Polynomially weak expansion alone does not supply this. For every fixed k, the Cartesian tori $C_L^{\square k}$ have bounded metric doubling, exact cop number $k+1$, and linear one-shot occupation cost at every radius. Taking $k > 1/\delta$ puts these examples inside every window $h(G) \geq |G|^{-\delta}$. A separate cubic replacement retains the barrier at $\delta = 1/2$. The obstacle in the universal problem is therefore not degree reduction itself; it is the need for adaptive reuse over many weak-growth layers.

2. Scale-adaptive cores and the robustness window

For a connected graph J, write

$$h(J) = \min_{\varnothing \neq A \subseteq V(J),\ |A| \leq |J|/2} \frac{|\partial_J A|}{|A|}.$$

The following elementary reduction explains why polynomially weak expansion is the relevant robustness window for the universal problem.

Proposition 2.1 (Scale-adaptive induced core)**.** *Let J be a connected graph of order n, and let $\eta(1) \geq \cdots \geq \eta(n) \geq 0$. Then J contains a connected induced subgraph K, of order m, such that*

$$\boxed{h(K) \geq \eta(m)} \qquad \text{and} \qquad \boxed{c(J) \leq c(K) + \sum_{j=m+1}^{n} \eta(j).}$$

Proof. Maintain the connected induced region J_i containing the robber. Whenever $|J_i| = m_i$ and $h(J_i) < \eta(m_i)$, choose $A_i \subseteq V(J_i)$ with $0 < |A_i| \leq m_i/2$ and $|\partial_{J_i} A_i| < \eta(m_i)|A_i|$, and occupy its boundary. The robber is then confined to one component J_{i+1} of $J_i - \partial_{J_i} A_i$. Whether that component lies inside A_i or outside it, one has

$$|A_i| \leq m_i - m_{i+1}.$$

Consequently the separator costs at most

$$\eta(m_i)(m_i - m_{i+1}) \le \sum_{j=m_{i+1}+1}^{m_i} \eta(j).$$

These integer intervals are disjoint along the robber's nested component chain. The process terminates at a connected induced K with $h(K) \ge \eta(|K|)$, and the separator costs telescope to the displayed sum. □

Taking $\eta(j) = j^{-a}$ shows that a polynomial cop-number saving on subcubic graphs with $h(K) \ge |K|^{-a}$ would imply a weak form of Meyniel for arbitrary graphs after the bounded-degree transfer of Hosseini–Mohar–Gonzalez Hermosillo de la Maza [7, Corollary 8]. Bradshaw–Hosseini–Mohar–Stacho already treat constant expansion restricted to sublinear set scales [3]; the unresolved axis in this reduction is expansion that itself shrinks polynomially.

3. Coarse occupation projections

Definition 3.1 (Coarse occupation projection)**.** Let G and H be connected graphs. A surjection $\pi : V(H) \to V(G)$ is a (λ, P)-occupation projection if, writing $F_v = \pi^{-1}(v)$,

(i) $|F_v| \le P$ for every $v \in V(G)$;

(ii) for distinct $u, v \in V(G)$ and arbitrary $x \in F_u$, $y \in F_v$,

$$\lambda(\mathrm{dist}_G(u,v) - 1) + 1 \le \mathrm{dist}_H(x,y) \le \lambda(\mathrm{dist}_G(u,v) + 2) - 2;$$

(iii) $\mathrm{diam}_H(F_v) \le 2(\lambda - 1)$ for every $v \in V(G)$.

The particular constants in Definition 3.1 are chosen because they are exact for the HMGHM tower. The proof below only needs a bounded additive slack after division by λ and a strict gap between deployment and escape deadlines.

For $U \subseteq V(G)$, let $B_G(U, r)$ be its closed radius-r neighborhood.

Theorem 3.2 (Abstract macro-ball occupation transfer)**.** *Let G have N vertices. Fix $d \ge 2$, $R \ge 2$, and constants $a, A_0 > 0$. Assume*

$$\sqrt{N} \le d^{R-2} < d\sqrt{N}, \qquad d^3 \le \sqrt{N}, \tag{1}$$

and

$$|B_G(U, R-2)| \ge a \min\{|U| d^{R-2}, N\} \qquad \text{for every } U \subseteq V(G), \tag{2}$$

$$|B_G(v, R)| \le A_0 d^R \qquad \text{for every } v \in V(G). \tag{3}$$

If H admits a (λ, P)-occupation projection onto G, then

$$\boxed{c(H) \le C(a, A_0) P\big(d^3 + \log(ePN)\big)\sqrt{N}.}$$

The displayed number of cops captures the robber within at most λR cop moves. In particular, the cop-number bound is independent of the scale factor λ; tower depth affects capture time but not the required bank size.

Proof. Put

$$\Theta = d^3 + \log(ePN), \qquad \mu = \frac{AP\Theta}{\sqrt{N}},$$

where A is a sufficiently large constant depending only on a, A_0. Choose one canonical vertex $z^* \in F_z$ in every fiber. At z^* place an independent Poisson number of cop tokens of mean μ.

For a possible robber fiber F_v, set

$$X_v = \pi^{-1}(B_G(v, R)).$$

By (3) and (1),

$$Q_v := |X_v| \le PA_0 d^R < A_0 P d^3 \sqrt{N} \le A_0 P \Theta \sqrt{N}. \tag{4}$$

We prove simultaneously for every v that the sampled tokens can be matched distinctly to all vertices of X_v, with every assigned token based over a base vertex within distance $R-2$ of its target fiber.

Let $S \subseteq X_v$, $|S| = s$, and put $U = \pi(S)$. Since every fiber has at most P targets, $|U| \ge s/P$. Every token based over $Z = B_G(U, R-2)$ is adjacent in the assignment graph to at least one target in S.

If $|U|d^{R-2} < N$, then the number of available tokens is Poisson with mean at least

$$\mu a |U| d^{R-2} \ge Aa\Theta s.$$

For a Poisson variable Y of mean $\Lambda \ge Aa\Theta s$,

$$\Pr(Y < s) \le e^{-\Lambda} \left(\frac{e\Lambda}{s}\right)^s.$$

After increasing A, this is at most $(ePN)^{-4s}$. There are at most $N\binom{PN}{s}$ choices of a root and a target subset of size s, so summing over $s \ge 1$ gives $o(1)$.

If $|U|d^{R-2} \ge N$, then the available-token mean is at least

$$\mu a N = AaP\Theta\sqrt{N} \ge \frac{Aa}{A_0} Q_v$$

by (4). Taking A large and applying the same Poisson bound shows, after a union bound over at most $N2^{Q_v}$ target subsets, that no saturated Hall condition fails with probability $1 - o(1)$. Here every saturated target set has

$$s \ge |U| \ge \frac{N}{d^{R-2}} > \frac{\sqrt{N}}{d} \ge N^{1/3},$$

so the polynomial prefactors are negligible.

Hall's condition therefore holds simultaneously for all macro-balls with probability $1 - o(1)$. The total number of sampled tokens is at most $2AP\Theta\sqrt{N}$ with probability $1 - o(1)$, so a deterministic placement of the claimed size exists.

It remains to compare deadlines. A target $y \in X_v$ lies over some $u \in B_G(v, R)$. Its assigned cop begins over z with $\mathrm{dist}_G(z, u) \le R - 2$. If $z \ne u$, the upper distortion bound gives travel time at most

$$\lambda((R-2)+2) - 2 = \lambda R - 2.$$

If $z = u$, the fiber-diameter bound gives at most $2(\lambda - 1) \le \lambda R - 2$, since $R \ge 2$.

To leave X_v, the robber must enter a fiber over a base vertex at distance at least $R+1$ from v. The lower distortion bound makes this require at least

$$\lambda((R+1)-1) + 1 = \lambda R + 1$$

steps. Every vertex of X_v is occupied first, and the cop assigned to the robber's current vertex captures her. □

Remark 3.3 (The quantifier needed from the random base)**.** The use of (2) is graph-uniform, not a per-source-set probability statement. In the dense theorem of Prałat and Wormald, condition (i) of their deterministic Theorem 3.1 is explicitly quantified over *every* source set and radius. Their Theorem 3.4 proves that a single $G(N, p)$ satisfies those hypotheses asymptotically almost surely; its proof unions over the bad source sets and concludes that the growth estimate holds simultaneously for all sets and radii [11, Theorems 3.1 and 3.4]. Thus the external input has the quantifier order required by Theorem 3.2.

4. The HMGHM replacement tower

For a vertex of degree r, the HMGHM replacement has one external port for every incident edge. The ports are partitioned into nearly equal classes, and for each pair of classes there is an internal vertex adjacent to every port in the two classes [7, Section 2].

Lemma 4.1 (One-round port geometry)**.** *For every HMGHM replacement cloud of degree at least two:*

(i) *distinct ports are nonadjacent and have distance exactly two;*
(ii) *every cloud vertex is within distance at most three of every specified port;*
(iii) *the cloud diameter is at most four.*

Proof. Two ports in different classes share the internal vertex associated with their class pair. Two ports in the same class share any internal vertex associated with that class and another nonempty class. Since ports are mutually nonadjacent, their distance is exactly two.

An internal vertex is adjacent to every port in either of two classes. If a specified port lies in neither class, travel to a port in one of the two classes, then through the internal vertex corresponding to that class and the specified port's class, and finally to the specified port. This takes three steps. The diameter bound follows by routing arbitrary endpoints through a specified port. □

Let

$$G = G_0, G_1, \dots, G_k = H$$

be an iterated HMGHM tower, and let $\pi : V(H) \to V(G)$ map every final vertex to its original ancestor. Put

$$F_v = \pi^{-1}(v), \qquad P = \max_v |F_v|, \qquad \lambda = 3^k.$$

Theorem 4.2 (Exact normalized distortion)**.** *The ancestry projection is a $(3^k, P)$-occupation projection. Explicitly, for distinct base vertices u, v, $r = \mathrm{dist}_G(u, v)$, and arbitrary $x \in F_u$, $y \in F_v$,*

$$\boxed{3^k(r-1) + 1 \le \mathrm{dist}_H(x, y) \le 3^k(r+2) - 2,}$$

and

$$\boxed{\mathrm{diam}_H(F_v) \le 2(3^k - 1).}$$

Proof. For one round, a shortest path between distinct clouds uses $e \ge r$ external edges. Between consecutive external edges it enters and leaves an intermediate cloud through distinct ports: otherwise it immediately traverses one external edge back. By Lemma 4.1, each intermediate port change costs at least two internal edges, and hence

$$\mathrm{dist}_{G_1}(x, y) \ge e + 2(e - 1) \ge 3r - 2.$$

For the upper bound, follow a base geodesic. Reaching the first prescribed port costs at most three, each intermediate port change costs two, the external edges cost r, and reaching the final endpoint costs at most three. Thus

$$\mathrm{dist}_{G_1}(x, y) \le 3 + r + 2(r - 1) + 3 = 3r + 4.$$

The one-round fiber diameter is at most four.

The lower and upper affine recurrences are

$$L_j(r) = 3L_{j-1}(r) - 2, \qquad U_j(r) = 3U_{j-1}(r) + 4,$$

with $L_0(r) = U_0(r) = r$. Solving gives

$$L_k(r) = 3^k(r - 1) + 1, \qquad U_k(r) = 3^k(r + 2) - 2.$$

The diameter recurrence $D_j \le 3D_{j-1} + 4$, $D_0 = 0$, gives $D_k \le 2(3^k - 1)$. □

The feature that matters is not the number of rounds but the normalized additive error: after division by 3^k, it remains two quotient layers. By Theorem 3.2, any other graph projection with the same three properties inherits the same occupation-certificate transfer.

5. Expansion retention under port-cloud replacement

Proposition 5.1 (Expansion under connected port replacement)**.** *Let H be obtained from a base graph G by replacing every vertex by a connected cloud of order at most L_0, with distinct external ports for the incident base edges. If $\iota(G)$ is the edge-isoperimetric constant of G, then*

$$\boxed{\iota(H) \geq \frac{1}{2L_0} \min\left\{1, \frac{\iota(G)}{L_0}\right\}.}$$

Proof. Let $S \subseteq V(H)$ with $0 < |S| \leq |H|/2$. In each cloud, classify the majority side and let $\mathcal{M}$ be the total number of minority vertices. Since every partially cut cloud is connected and has at most L_0 vertices, its internal cut contributes at least one edge, so the total internal contribution is at least $\mathcal{M}/L_0$.

Let U be the set of base vertices whose clouds have majority in S, and put $E = e_G(U, V(G) \setminus U)$. A base cut edge can fail to cross the lifted cut only if one of its two ports is a minority vertex. Distinct base edges use distinct ports, so at most $\mathcal{M}$ of the E external cut edges fail. Hence

$$e_H(S, V(H) \setminus S) \geq \frac{\mathcal{M}}{L_0} + \max\{0, E - \mathcal{M}\} \geq \frac{E + \mathcal{M}}{2L_0}.$$

Apply the same majority accounting to S or its complement, according as $|U| \leq |G|/2$ or not, to obtain

$$\min\{|U|, |V(G) \setminus U|\} \geq \frac{|S| - \mathcal{M}}{L_0}.$$

Thus

$$E \geq \frac{\iota(G)}{L_0}(|S| - \mathcal{M}),$$

and consequently

$$E + \mathcal{M} \geq \min\left\{1, \frac{\iota(G)}{L_0}\right\} |S|.$$

Combining the inequalities proves the proposition. □

The HMGHM cloud-size recurrence turns the one-round estimate into a subpolynomial-loss statement in the polylogarithmic-degree regime. If D is the initial maximum degree, L_i is the largest cloud order in round i, and k rounds are used, HMGHM prove

$$\prod_{i=0}^{k-1} L_i \leq C D^2 (\log D)^{\log_2(11/5)}, \qquad 2^k = O(\log D).$$

Iterating Proposition 5.1 therefore gives the following.

Corollary 5.2 (Expansion retained by HMGHM reduction)**.** *Let H be the final subcubic graph obtained from a connected graph G of maximum degree $D \geq 4$. Then*

$$\boxed{h(H) \geq \frac{\iota(G)}{C D^4 (\log D)^{\kappa}}, \qquad \kappa = 1 + 2\log_2(11/5) < 3.28.}$$

In particular, if $D = |G|^{o(1)}$ and $\iota(G) = |G|^{-o(1)}$, then $|H| = |G|^{1+o(1)}$ and $h(H) = |H|^{-o(1)}$.

Proof. At every round $\iota(G_i) \le D_i \le L_i$, so Proposition 5.1 gives

$$\iota(G_{i+1}) \ge \frac{\iota(G_i)}{2L_i^2}.$$

Thus

$$\iota(H) \ge \frac{\iota(G)}{2^k(\prod_i L_i)^2} \ge \frac{\iota(G)}{CD^4(\log D)^\kappa}.$$

Since H has maximum degree at most three, its vertex expansion is at least one third of its edge expansion. The order statement follows from the same cloud-product bound. □

Remark 5.3 (Relation to replacement products)**.** The regular replacement-product literature proves stronger spectral conclusions under much stronger hypotheses on the clouds; see, for example, Reingold–Vadhan–Wigderson [12]. Proposition 5.1 allows arbitrary connected, nonuniform clouds and consequently gives only a crude isoperimetric estimate. No novelty claim is made here beyond this precise form without a fuller graph-substitution review.

6. A polylogarithmically tight square-root family

Take

$$d = (\log N)^4, \qquad p = \frac{d}{N-1}, \qquad G \sim G(N,p).$$

Iterate the HMGHM replacement until the graph H is subcubic, and write $M = |H|$.

With high probability, $\Delta(G) \le 2d$. By Remark 3.3, the dense Prałat–Wormald theorem supplies the uniform lower growth in (2); in the volume range used here it also supplies the upper growth in (3) [11]. Choose R minimally so that $d^{R-2} \ge \sqrt{N}$. Since d is polylogarithmic, (1) holds.

HMGHM give, both globally and along one ancestry fiber,

$$P \le Cd^2(\log d)^{1.14}, \qquad N \le M \le PN. \tag{5}$$

Their shadow strategy gives $c(H) \ge c(G)$, and the random-graph lower bound of Bollobás–Kun–Leader used in their argument yields

$$c(G) \ge d^{-2} N^{\frac{1}{2} - \frac{9}{2\log\log d}}.$$

The abstract transfer theorem gives

$$c(H) \le CP\big(d^3 + \log(ePN)\big)\sqrt{N} \le \sqrt{M}\,(\log M)^{20+o(1)}.$$

Using $d = (\log N)^4$ and $M = N(\log N)^{O(1)}$ in the lower bound gives the following more informative formulation.

Theorem 6.1 (Quantitative HMGHM hard family)**.** *There is a sequence of connected subcubic graphs H, of order $M \to \infty$, for which*

$$\boxed{M^{\frac{1}{2} - \frac{9+o(1)}{2\log\log\log M}} \le c(H) \le \sqrt{M}\,(\log M)^{20+o(1)}.}$$

The upper bound is $\sqrt{M}$ times a polylogarithmic factor. The lower exponent tends to $1/2$ only at a triple-logarithmic rate.

The base edge expansion is $\Omega(d)$ with high probability. Since $d = \operatorname{polylog} N$, Corollary 5.2 gives

$$h(H) \ge (\log M)^{-O(1)} = M^{-o(1)}.$$

This is exactly the degree regime in which the retention factor is informative; for polynomial initial degree the crude D^4 loss can be vacuous.

Corollary 6.2 (Square-root weak-expander family)**.** *There are connected subcubic graphs satisfying*

$$h(H) \geq M^{-o(1)} \qquad \text{and} \qquad c(H) = M^{1/2+o(1)}.$$

More precisely, they obey the two-sided bounds of Theorem 6.1.

Remark 6.3 (What is forced, and what is achieved)**.** By Theorem 2.1, polynomially weak expansion is a natural robustness window for weak Meyniel. The HMGHM lower bound alone already forces every proposed estimate

$$c(J) \leq C\phi^{-p}|J|^{1-\varepsilon+o(1)} \qquad (h(J) \geq \phi = |J|^{-o(1)})$$

to have $\varepsilon \leq 1/2$. That restriction predates the upper transfer proved here. The new conclusion is that the known stressing family itself achieves the endpoint order $\sqrt{M}$ up to polylogarithmic factors, so this family is not an obstruction to a Meyniel-strength theorem on polynomially weak subcubic expanders.

7. Why chase strategies need not transfer

The abstract theorem deliberately transfers a strategy class, not arbitrary cop number. The smallest example explains the distinction. For a degree-two vertex, one HMGHM cloud is a three-vertex path. Replacing every vertex of C_3 therefore produces C_9. But

$$c(C_3) = 1, \qquad c(C_9) = 2.$$

The one-cop win on C_3 is a direct chase/dismantling phenomenon. The subdivision-like stretching destroys it. By contrast, an occupation certificate is synchronized to a deadline: the replacement tower stretches the cops' travel and the robber's escape by the same factor, and the bounded normalized additive slack preserves a strict margin. The examples K_4 and the diamond graph exhibit the same one-round increase, so the issue is structural rather than peculiar to one cycle.

8. A one-shot occupation barrier

Definition 8.1 (Universal one-shot occupation number)**.** For a connected graph G and integer $R \geq 0$, let $\mathrm{Occ}_R(G)$ be the minimum size of a finite set X of distinct cop tokens, equipped with a position map $p : X \to V(G)$, such that for every $v \in V(G)$ there is an injection

$$f_v : B_G(v, R) \longrightarrow X$$

with

$$\mathrm{dist}_G(u, p(f_v(u))) \leq R \qquad \text{for every } u \in B_G(v, R).$$

Different tokens may have the same initial position. After learning the robber's starting vertex, the common prepositioned bank can occupy her entire radius-R ball within R moves.

Remark 8.2 (Why the target and deadline are natural)**.** If the robber starts at v, she needs at least $R+1$ robber moves to leave $B_G(v, R)$. Occupying that whole ball within R cop moves is therefore the canonical one-shot certificate: every vertex she could still occupy is filled before her first possible escape. The parameter Occ_R measures this specific strategy class, not ordinary cop number.

Theorem 8.3 (Counting barrier)**.** *Every connected graph satisfies*

$$\boxed{\mathrm{Occ}_R(G) \geq \frac{\sum_{v \in V(G)} |B_G(v, R)|}{\max_{x \in V(G)} |B_G(x, 2R)|}.}$$

In particular, if G is vertex-transitive, then

$$\boxed{\mathrm{Occ}_R(G) \geq |V(G)| \frac{|B_G(o, R)|}{|B_G(o, 2R)|}.}$$

Proof. Fix a feasible multiset X. For each possible robber start v, every cop token used by the injection f_v lies in $B_G(v, 2R)$, by the triangle inequality. Hence at least $|B_G(v, R)|$ tokens of X lie in $B_G(v, 2R)$. Summing over v, the number of incident pairs (v, x) with $x \in X \cap B_G(v, 2R)$ is at least $\sum_v |B_G(v, R)|$.

A fixed token based at x is counted only for starts $v \in B_G(x, 2R)$, at most $\max_y |B_G(y, 2R)|$ times. Therefore

$$|X| \max_y |B_G(y, 2R)| \geq \sum_v |B_G(v, R)|,$$

which proves the claim. □

The theorem identifies the exact growth ratio demanded by one-shot occupation. A polynomial saving from the trivial $|V(G)|$ bound requires polynomial amplification from radius R to radius $2R$.

We now give subcubic witnesses showing that polynomially weak expansion does not imply such amplification.

Definition 8.4 (The cubic truncated torus)**.** For $L \geq 5$, let Q_L have vertex set

$$(\mathbb{Z}/L\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}.$$

Inside each fiber $(x, y) \times \mathbb{Z}/4\mathbb{Z}$, join the four vertices in a cycle. Add the external edges

$$(x, y, 0)(x, y + 1, 2) \qquad \text{and} \qquad (x, y, 1)(x + 1, y, 3)$$

for every (x, y). Equivalently, $(x, y, 2)$ receives its external edge from $(x, y - 1, 0)$, and $(x, y, 3)$ receives its external edge from $(x - 1, y, 1)$. Thus every vertex has two internal cycle neighbors and one external neighbor, and Q_L is the four-cycle port replacement of the square torus $C_L \square C_L$.

Every vertex of Q_L has degree three, and the construction embeds on the torus by replacing each base vertex inside a small disk. Its order is $4L^2$.

Lemma 8.5 (Vertex transitivity and explicit doubling of Q_L)**.** *The graph Q_L is vertex-transitive and, for every vertex x and radius $R \geq 0$,*

$$|B_{Q_L}(x, 2R)| \leq 5500\, |B_{Q_L}(x, R)|.$$

Proof. Translations in the first two coordinates are automorphisms. The map

$$\rho(x, y, i) = (y, -x, i + 1)$$

(with coordinates interpreted cyclically) preserves internal cycle edges and interchanges the two external edge directions. Translations together with ρ act transitively.

Let $T_L = C_L \square C_L$ and project (x, y, i) to (x, y). Projection does not increase distance, so

$$|B_{Q_L}(x, 2R)| \leq 4|B_{T_L}(\pi x, 2R)| \leq 4 \min\{L, 4R + 1\}^2.$$

Conversely, from an arbitrary cloud vertex one can enter the required port in at most two internal moves and then lift each base step using at most three moves. Hence, with $r = \lfloor (R - 2)/3 \rfloor$ for $R \geq 2$,

$$|B_{Q_L}(x, R)| \geq |B_{T_L}(\pi x, r)|.$$

The coordinate box of cyclic radius $\lfloor r/2 \rfloor$ lies inside the ℓ_1 ball, so

$$|B_{T_L}(\pi x, r)| \geq \min\{L, 2\lfloor r/2 \rfloor + 1\}^2.$$

For $R < 10$, the upper bound is at most $4 \cdot 37^2 < 5500$ and the denominator is at least one. For $R \geq 10$, one has $r \geq R/6$ and $2\lfloor r/2 \rfloor + 1 \geq r$, whence the ratio is at most $4 \cdot 30^2 < 5500$. This proves the displayed constant. □

Theorem 8.6 (Cubic one-shot barrier)**.** *For $L \geq 5$, the connected cubic graphs Q_L, with $M = |Q_L| = 4L^2$, satisfy*

$$\boxed{h(Q_L) = \Theta(M^{-1/2}), \qquad c(Q_L) \leq 3, \qquad \mathrm{Occ}_R(Q_L) \geq \frac{M}{5500} \quad \text{for every } R \geq 0.}$$

Proof. The square torus has edge and vertex expansion $\Theta(1/L)$ by the discrete torus isoperimetric inequality [5]. Applying Proposition 5.1 with cloud size four gives the matching lower bound for Q_L; lifting a coordinate slab of width $\lfloor L/2 \rfloor$ gives the upper bound for every L. Since Q_L is cubic, edge and vertex expansion differ by at most a constant factor. Thus $h(Q_L) = \Theta(1/L) = \Theta(M^{-1/2})$.

The graph Q_L is toroidal, and every toroidal graph has cop number at most three [8]. Vertex transitivity, Theorem 8.3, and Lemma 8.5 give

$$\mathrm{Occ}_R(Q_L) \geq M \frac{|B_{Q_L}(x, R)|}{|B_{Q_L}(x, 2R)|} \geq \frac{M}{5500}.$$

The finite audit suggests that the optimal asymptotic constant is $1/4$, but that sharpening is not needed here. □

8.1. **The barrier throughout every polynomial expansion window.** For integers $k \geq 2$ and $L \geq 4$, write

$$T_{L,k} = \underbrace{C_L \square \cdots \square C_L}_{k \text{ factors}}.$$

Its order is $m = L^k$, its degree is $2k$, and its metric is the cyclic ℓ_1 metric.

Lemma 8.7 (Uniform doubling of Cartesian tori)**.** *For every fixed $k \geq 2$, every $L \geq 4$, every vertex x, and every radius $R \geq 0$,*

$$|B_{T_{L,k}}(x, 2R)| \leq (5k)^k |B_{T_{L,k}}(x, R)|.$$

Proof. Every coordinate of a point in $B(x, 2R)$ has cyclic distance at most $2R$, so

$$|B(x, 2R)| \leq \min\{L, 4R + 1\}^k.$$

The coordinate box in which every coordinate has cyclic distance at most $\lfloor R/k \rfloor$ lies in $B(x, R)$, and therefore

$$|B(x, R)| \geq \min\{L, 2\lfloor R/k \rfloor + 1\}^k.$$

If $R < k$, the ratio is at most $(4k + 1)^k$. If $R \geq k$, then $2\lfloor R/k \rfloor + 1 \geq R/k$ and $4R + 1 \leq 5R$; taking the minima with L does not increase their ratio beyond $5k$. The claim follows. □

Theorem 8.8 (Full-window toroidal barrier)**.** *For every fixed $k \geq 2$, the graphs $T_{L,k}$ satisfy*

$$\boxed{h(T_{L,k}) = \Theta_k(m^{-1/k}), \qquad c(T_{L,k}) = k + 1, \qquad \mathrm{Occ}_R(T_{L,k}) \geq (5k)^{-k} m \quad \text{for every } R \geq 0.}$$

Consequently, for every $\delta > 0$ there is a constant-degree graph family with

$$h(G) \geq |G|^{-\delta}, \qquad c(G) = O_\delta(1), \qquad \mathrm{Occ}_R(G) = \Omega_\delta(|G|) \quad \text{for every radius } R.$$

Proof. The discrete-torus edge-isoperimetric inequality gives order $1/L$ [5]. Since $T_{L,k}$ has degree $2k$, edge boundary and external vertex boundary differ by at most the fixed factor $2k$; a coordinate slab of width $\lfloor L/2 \rfloor$ supplies the matching upper bound. Hence $h(T_{L,k}) = \Theta_k(1/L) = \Theta_k(m^{-1/k})$. Neufeld and Nowakowski proved that a Cartesian product of k cycles, each of length at least four, has cop number exactly $k + 1$ [10]. Since the torus is vertex-transitive, Theorem 8.3 and Lemma 8.7 give the occupation lower bound.

Given $\delta > 0$, choose $k = \max\{2, \lfloor 1/\delta \rfloor + 1\}$. Then $1/k < \delta$, so for sufficiently large m the expansion lower bound $h(T_{L,k}) \geq m^{-\delta}$ holds after absorbing the fixed k-dependent constant. □

Remark 8.9 (The sharp metric constant)**.** For fixed k, choose radii $1 \ll R \ll L$. Lattice-point asymptotics for the ℓ_1 ball give

$$\frac{|B_{T_{L,k}}(x, 2R)|}{|B_{T_{L,k}}(x, R)|} = 2^k + o(1).$$

Thus no uniform doubling constant below 2^k is possible, and the counting bound of Theorem 8.3 approaches the natural fraction $2^{-k}m$ on these local radii. The explicit constant $(5k)^{-k}$ is chosen only for a short all-radii proof.

Corollary 8.10 (No expansion-only one-shot theorem)**.** *For every $\delta > 0$, there is no implication of the form*

$$h(G) \geq |G|^{-\delta} \quad \Longrightarrow \quad \mathrm{Occ}_R(G) \leq |G|^{1-\varepsilon} \text{ for some radius } R$$

with any fixed $\varepsilon > 0$, even when the maximum degree is bounded by a constant depending only on δ.

Proof. Take the family from Theorem 8.8. It lies in the prescribed expansion window, while $\mathrm{Occ}_R(G) = \Omega_\delta(|G|)$ for every R. □

Remark 8.11 (Architectural meaning)**.** The logical obstruction is not that tori are difficult pursuit instances; they are not. Rather, Theorem 8.3 makes every vertex-transitive bounded-doubling graph expensive for one-shot occupation, and bounded doubling is compatible with every polynomial weak-expansion window by Theorem 8.8. The tori certify that the hypothesis class contains such graphs while coordinate-wise shadowing still uses exactly $k + 1$ cops. Therefore ball amplification is the wrong invariant for adaptive pursuit, not merely a poor description of one particular easy family. The cubic family Q_L records that the same separation already occurs in maximum degree three at exponent $1/2$.

9. Outlook

For vertex expansion $h(G) \geq \phi$, iterating the elementary growth factor $1 + \phi$ reaches global scale after $O(\phi^{-1} \log |G|)$ layers. The same iteration gives the standard diameter bound of that order; these are two forms of the same calculation, not independent evidence. The strategic consequence is that, when $\phi = |G|^{-a}$, an amplification-based pursuit scheme must operate over a full-traversal timescale $\Theta(|G|^a \log |G|)$.

The present paper separates three phenomena:

(1) bounded normalized metric distortion preserves a strong occupation certificate through degree reduction;
(2) the HMGHM stressing family itself meets the square-root endpoint up to polylogarithmic factors;
(3) one-shot occupation is nevertheless incapable of proving a universal robustness theorem throughout any polynomial weak-expansion window, even on bounded-degree graphs with constant cop number; a cubic instance already appears at exponent $1/2$.

The remaining universal question is therefore an adaptive one. On the tori, ball growth carries essentially no information about pursuit cost; product structure instead supports coordinate-wise shadowing. What geometric or combinatorial quantity replaces product coordinates on a general polynomially weak expander? Equivalently, can a capacitated, correlated, or deferred witness system reuse the same cop resources over polynomially many weak-growth layers, or must every such one-traversal certificate incur polynomial congestion?

Acknowledgments

The author is grateful to Anthony Clow, Peter Bradshaw, Bojan Mohar, and Florian Lehner for work and perspectives that helped shape the questions addressed here. Additional acknowledgments

will be added in a later version. The author welcomes corrections concerning priority, related graph-substitution inequalities, and the scope of the occupation framework.

Audit and reproducibility

The metric inequalities were independently tested on HMGHM towers rebuilt from the published gadget description, including structured base graphs not used in the original audit. The Hall inequalities and timing margins were checked numerically, and exact small replacement games were solved by retrograde analysis. The toroidal barrier audit computes exact ball profiles of $C_L^{\square k}$ by convolving cyclic distance distributions, verifies the $(5k)^k$ doubling bound for $k = 2, 3, 4, 5$, constructs Q_L, checks cubicity, connectivity, and the displayed rotation automorphism, and evaluates the counting lower bound at every radius. No theorem depends on the computations.

References

[1] M. Aigner and M. Fromme, *A game of cops and robbers*, Discrete Applied Mathematics 8 (1984), 1–12.
[2] P. Bose, L. Esperet, J. Hodor, G. Joret, P. Micek, and C. Rambaud, *Cops and robber in graphs with bounded vertex cover number*, arXiv:2602.07435, 2026.
[3] P. Bradshaw, S. A. Hosseini, B. Mohar, and L. Stacho, *On the cop number of graphs of high girth*, Journal of Graph Theory 102 (2023), 15–34; arXiv:2005.10849.
[4] B. Bollobás, G. Kun, and I. Leader, *Cops and robbers in a random graph*, Journal of Combinatorial Theory, Series B 103 (2013), 226–236.
[5] B. Bollobás and I. Leader, *An isoperimetric inequality on the discrete torus*, SIAM Journal on Discrete Mathematics 3 (1990), 32–37.
[6] A. Clow, *Expanders satisfy the weak Meyniel conjecture*, withdrawn preprint, arXiv:2311.13792, 2023.
[7] S. A. Hosseini, B. Mohar, and S. Gonzalez Hermosillo de la Maza, *Meyniel's conjecture on graphs of bounded degree*, Journal of Graph Theory 97 (2021), 401–407; arXiv:1912.06957.
[8] F. Lehner, *On the cop number of toroidal graphs*, Journal of Combinatorial Theory, Series B 151 (2021), 250–262.
[9] L. Lu and X. Peng, *On Meyniel's conjecture of the cop number*, Journal of Graph Theory 71 (2012), 192–205.
[10] S. Neufeld and R. Nowakowski, *A game of cops and robbers played on products of graphs*, Discrete Mathematics 186 (1998), 253–268.
[11] P. Prałat and N. Wormald, *Meyniel's conjecture holds for random graphs*, Random Structures & Algorithms 48 (2016), 396–421; arXiv:1301.2841.
[12] O. Reingold, S. Vadhan, and A. Wigderson, *Entropy waves, the zig-zag graph product, and new constant-degree expanders*, Annals of Mathematics 155 (2002), 157–187.
[13] A. Scott and B. Sudakov, *A bound for the cops and robbers problem*, SIAM Journal on Discrete Mathematics 25 (2011), 1438–1442.

Brown University